Phase transitions induced by noise cross–correlations

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A general approach to consider spatially extended stochastic systems with correlations between additive and multiplicative noises subject to nonlinear damping is developed. Within modified cumulant expansion method, we derive an effective Fokker–Planck equation whose stationary solutions describe a character of ordered state. We find that fluctuation cross–correlations lead to a symmetry breaking of the distribution function even in the case of the zero–dimensional system. In general case, continuous, discontinuous and reentrant noise induced phase transitions take place. It is appeared the cross–correlations play a role of bias field which can induce a chain of phase transitions being different in nature. Within mean field approach, we give an intuitive explanation of the system behavior through an effective potential of thermodynamic type. This potential is written in the form of an expansion with coefficients defined by temperature, intensity of spatial coupling, auto- and cross–correlation times and intensities of both additive and multiplicative noises.

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I. INTRODUCTION

A considerable activity concerning the stochastic processes addresses the constructive role of fluctuations of environment being called as noises. An incomplete list of such processes includes noise induced unimodal–bimodal transitions [1], stochastic resonance [2], noise induced spatial patterns and phase transitions [3], etc. In general case, considering a stochastic dynamics, one should deal with a problem to account for correlations between random sources. Along this line several special methods were developed. Most popular of them are as follows: (i) the cumulant expansion method [4, 5]; (ii) the spectral width expansion method [1, 6]; (iii) the unified colored noise approximation where evolution equations for both a stochastic variable and a random force are combined within unique equation of motion [7, 8, 9, 10].

The wide spectra of works are aimed to explore an effect of correlations of fluctuations in extended systems (see Ref.[3] and references therein). It is appeared the interplay among noise correlations, nonlinearity and spatial coupling leads to a special type of noise induced phase transition known as reentrant transition [11]. Moreover, such reentrance can be observed even in the limit of weakly correlated noise: if the noise self–correlation time $\tau \to 0$, the system undergoes a single reentrant phase transition [12]. In the opposite limit of the strong correlated noise ($\tau \to \infty$) a chain of reentrant phase transitions can take place [8]. A kind of these transitions is continuous or discontinuous in dependence of the self–action part of a bare potential.

Quite peculiar picture is observed in the case of several noises whose cross–correlations can arrive at the remarkable and counterintuitive phenomena related to the problem of reconstruction of phase transitions. In such a case the stochastic system undergoes a chain of different phase transitions with appearance of a metastable phase in spite of the fact that bare potential does not assume such a phase [13]. Unfortunately, nowadays we have a scanty collection of works devoted to this phenomenon. Perhaps, it can be explained by problems in realization of natural or computer experiments in extended systems, on the one hand, and by absence of theoretical tools and methods to perform the corresponding calculations, on the other hand.

Above pointed phenomena force to reconsider before developed ideas concerning the theoretical approaches of noise induced phase transitions in extended systems. Most articles concern the problem of above cross–correlations within framework of special models which are difficult to generalize for description of reconstruction of phase transitions. Therefore, the fluctuation induced rebuilding in the system behavior is an open question that should be developed to find the unique role of stochastic environment influence.

In this paper, we consider the general situation to present the role of cross-correlation contribution of two noises into the picture of phase transitions. We explore an extended stochastic system which obeys the archetypal model of Brownian particle. The adequate scheme which allows to specify statistical properties of the system with nonlinear kinetic coefficient in the overdamped limit is introduced. Within a simplest model with nonlinear damping, drift caused by Landau–like potential and two colored (multiplicative and additive) noises, we show in what a way the system can undergo noise induced phase transitions. We find out phase transitions of both continuous and discontinuous character are realized as biased phase transitions. We obtain phase and bifurcation diagrams to elucidate the cross–correlation is one of reasons to appear the metastable states inherent in first order phase transitions.

The paper is organized in the following manner. Section II is devoted to development of analytical ap-

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proach to study noise induced phase transitions on the basis of kinetic equation for the probability density function and cumulant expansion method. In Section III, we apply the derived formalism to consider noise induced phase transitions in the simplest case of the Ginzburg–Landau model with both multiplicative and additive noises and kinetic coefficient dependent of the stochastic variable. On the basis of obtained drift and diffusion coefficients we build out an effective one–dimensional stochastic process being multiplicative but noncorrelated. Related probability distribution function combined with self–consistency condition allow us to investigate the corresponding phase diagram and stationary behavior of the order parameter (Section IV). Discussion in Section V is based on representation of the effective stochastic process within framework of the mean field approach. Such consideration allows us to study noise induced phase transitions by analogy with the standard Landau scheme. It is appeared the noise cross–correlations and strengthening dispersion of damping coefficient arrive at transformation of the continuous phase transition into discontinuous one. Finally, main results and perspectives of the work are collected in Conclusion (Section VI).

II. MAIN RELATIONS

Consider a Brownian particle under influence of an effective potential $\mathcal{F}[x(\mathbf{r})]$ and a damping characterized by the viscosity coefficient $\gamma(x)$. The generalized evolution equations for the scalar variable x and the conjugate momentum p read:

$$m\dot{x} = p, \tag{1}$$

$$\dot{p} + \gamma(x)p = -\frac{\delta\mathcal{F}}{\delta x(\mathbf{r},t)} + g_\mu(x)\zeta_\mu(\mathbf{r},t) \tag{2}$$

where m is the effective particle mass, dot stands for the derivative with respect to the time t, $\mathbf{r}$ is the space coordinate; the effective potential is reduced to the Ginzburg–Landau form

$$\mathcal{F} = \int \left[V_0(x) + \frac{D}{2}|\nabla x|^2\right] \mathrm{d}\mathbf{r} \tag{3}$$

with $V_0(x)$ and $D > 0$ being a specific thermodynamic potential and an inhomogeneity constant, $\nabla \equiv \partial/\partial\mathbf{r}$. The last term in Eq.(2), where index $\mu = 1, 2$ numerates different noises to be summarized in accordance with the Einstein rule, represents Langevin forces which act with amplitudes $g_\mu(x)$ and stochastically alternating functions ζ_μ. Neglecting a space correlation, we focus on time correlations between forces ζ_μ, i.e.,

$$\langle\zeta_\mu(\mathbf{r},t)\zeta_\nu(\mathbf{r}',t')\rangle = \delta(\mathbf{r}-\mathbf{r}')C_{\mu\nu}(t-t'). \tag{4}$$

Inserting Eq.(2) into the result of differentiating Eq.(1) over time, it is convenient to represent evolution equation for the quantity x in the form

$$m\ddot{x} + \gamma(x)\dot{x} = f(x) + D\Delta x + g_\mu(x)\zeta_\mu \tag{5}$$

where $f(x) = -\mathrm{d}V_0/\mathrm{d}x$ is a deterministic force.

To study statistical properties of the system one needs to find the probability density function $P = P(p, x, t)$ of the system states distribution in the phase state $\{x, p\}$. With this aim, we represent the system on the regular d–dimension lattice with mesh size $\Delta\ell = 1$ and $\gamma_i = \gamma(x_i)$, $f_i = f(x_i)$, $g_{\mu i} = g_\mu(x_i)$. Then, the differential equation with partial derivatives (5) is reduced to the usual differential equation

$$\ddot{x}_i + \gamma_i\dot{x}_i = f_i + \frac{D}{2d}\sum_j \widehat{D}_{ij}x_j + g_{\mu i}\zeta_{\mu i} \tag{6}$$

due to representation of the Laplacian operator on a grid as follows

$$\Delta \to \Delta_i \equiv \sum_j \widehat{D}_{ij} = \sum_j (\delta_{nn(i),j} - 2d\delta_{ij}) \tag{7}$$

where $nn(i)$ notices a set of the nearest neighbors of the site i.

By definition, the probability density function is given by averaging over noises of the density function $\rho(x_i, p_i, t)$ of the microscopic states distribution in the phase space:

$$P(x_i, p_i, t) = \langle\rho(x_i, p_i, t)\rangle. \tag{8}$$

To construct an equation for the macroscopic density function $P = P(x_i, p_i, t)$ we exploit the conventional device to proceed from the continuity equation for the microscopic one $\rho = \rho(x_i, p_i, t)$:

$$\frac{\partial\rho}{\partial t} + \left[\frac{\partial}{\partial x_i}(\dot{x}_i\rho) + \frac{\partial}{\partial p_i}(\dot{p}_i\rho)\right] = 0. \tag{9}$$

Inserting the time derivative of the momentum $\dot{p} = m\ddot{x}$ from Eq.(6) into Eq.(9), we obtain

$$\frac{\partial\rho}{\partial t} = \left(\widehat{\mathcal{L}} + \widehat{\mathcal{N}}_\mu\zeta_\mu\right)\rho \tag{10}$$

where the operators $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{N}}_\mu$ are defined as follows:

$$\widehat{\mathcal{L}} \equiv -\frac{p_i}{m}\frac{\partial}{\partial x_i} - \frac{\partial}{\partial p_i}\left(f_i + \frac{D}{2d}\sum_j \widehat{D}_{ij}x_j - \frac{\gamma_i}{m}p_i\right), \tag{11}$$

$$\widehat{\mathcal{N}}_\mu \equiv -g_{\mu i}\frac{\partial}{\partial p_i}. \tag{12}$$

Within the interaction representation, the microstate density function reads as

$$\wp = e^{-\widehat{\mathcal{L}}t}\rho \tag{13}$$

to reduce Eq.(10) to the form

$$\frac{\partial\wp}{\partial t} = \sum_\mu \widehat{\mathcal{R}}_\mu \wp, \tag{14}$$

$$\widehat{\mathcal{R}}_\mu = \widehat{\mathcal{R}}_\mu(x_i, p_i, t) \equiv \zeta_\mu \left(e^{-\widehat{\mathcal{L}}t}\widehat{\mathcal{N}}_\mu e^{\widehat{\mathcal{L}}t}\right). \tag{15}$$

A standard and effective device to solve such a type of stochastic equation is the well–known cumulant expansion method [4]. Neglecting terms of the order $O(\widehat{\mathcal{R}}_\mu^3)$, we arrive at the kinetic equation of the form

$$\frac{\partial}{\partial t}\langle\wp\rangle(t) = \left[\sum_{\mu\nu}\int\limits_0^t \langle\widehat{\mathcal{R}}_\mu(t)\widehat{\mathcal{R}}_\nu(t')\rangle \mathrm{d}t'\right]\langle\wp\rangle(t). \tag{16}$$

Within the original representation, the equation for the probability density (8) reads

$$\frac{\partial}{\partial t}P(t) = \left\{\widehat{\mathcal{L}} + \int\limits_0^t C_{\mu\nu}(\tau)\left[\widehat{\mathcal{N}}_\mu \left(e^{\widehat{\mathcal{L}}\tau}\widehat{\mathcal{N}}_\nu e^{-\widehat{\mathcal{L}}\tau}\right)\right]\mathrm{d}\tau\right\}P(t). \tag{17}$$

If the physical time is much more than a correlation scale ($t \gg \tau_\mu$), we can replace the upper limit of the integration by $t = \infty$. Then, expanding exponents, we derive to the perturbation expansion

$$\frac{\partial P}{\partial t} = \left(\widehat{\mathcal{L}} + \widehat{\mathcal{C}}\right)P \tag{18}$$

where collision operator

$$\widehat{\mathcal{C}} \equiv \sum_{n=0}^{\infty}\widehat{\mathcal{C}}^{(n)}, \quad \widehat{\mathcal{C}}^{(n)} \equiv M_{\mu\nu}^{(n)}\left(\widehat{\mathcal{N}}_\mu\widehat{\mathcal{L}}_\nu^{(n)}\right) \tag{19}$$

is determined through the commutators

$$\widehat{\mathcal{L}}_\nu^{(n+1)} = [\widehat{\mathcal{L}}, \widehat{\mathcal{L}}_\nu^{(n)}], \quad \widehat{\mathcal{L}}_\nu^{(0)} \equiv \widehat{\mathcal{N}}_\nu \tag{20}$$

and moments of correlation function

$$M_{\mu\nu}^{(n)} = \frac{1}{n!}\int_0^\infty \tau^n C_{\mu\nu}(\tau)\mathrm{d}\tau. \tag{21}$$

To perform next calculations we shall restrict ourselves considering overdamped systems where the variation scales t_s, x_s, v_s, γ_s, f_s, and g_s of the time t, the quantity x, the velocity $v \equiv p/m$, the damping coefficient $\gamma(x)$, the force $f(x)$ and the noise amplitudes $g_\mu(x)$, respectively, obey the following conditions:

$$\begin{aligned}
&\frac{v_s t_s}{x_s} \equiv \epsilon^{-1} \gg 1, \quad \gamma_s t_s = \epsilon^{-2} \ggg 1,\\
&\frac{f_s t_s}{v_s m} = \epsilon^{-1} \gg 1, \quad \frac{g_s t_s}{v_s m} = \epsilon^{-1} \gg 1.
\end{aligned} \tag{22}$$

These conditions means a hierarchy of the damping and the deterministic/stochastic forces to be characterized by relations

$$\frac{f_s/m}{\gamma_s v_s} = \epsilon \ll 1, \quad \frac{g_s/m}{\gamma_s v_s} = \epsilon \ll 1. \tag{23}$$

As a result, the dimensionless system of equations (1), (2) takes the form

$$\begin{aligned}
\frac{\partial x}{\partial t} &= \epsilon^{-1}v,\\
\frac{\partial v}{\partial t} &= -\epsilon^{-2}\gamma(x)v\\
&\quad + \epsilon^{-1}\left[f(x) + D\Delta x + g_\mu(x)\zeta_\mu(\mathbf{r}, t)\right].
\end{aligned} \tag{24}$$

Respectively, the Fokker–Planck equation (18) reads

$$\left(\frac{\partial}{\partial t} - \widehat{\mathcal{L}}\right)P = \epsilon^{-2}\widehat{\mathcal{C}}P \tag{25}$$

where the operator

$$\widehat{\mathcal{L}} \equiv \epsilon^{-1}\widehat{\mathcal{L}}_1 + \epsilon^{-2}\widehat{\mathcal{L}}_2 \tag{26}$$

has the components

$$\begin{aligned}
&\widehat{\mathcal{L}}_1 \equiv -v_i\frac{\partial}{\partial x_i} - \left(f_i + \frac{D}{2d}\sum_j \widehat{D}_{ij}x_j\right)\frac{\partial}{\partial v_i},\\
&\widehat{\mathcal{L}}_2 \equiv \gamma_i\,\frac{\partial}{\partial v_i}v_i.
\end{aligned} \tag{27}$$

The collision operator $\widehat{\mathcal{C}}$ is defined through the expressions (19) — (21) and the operator (12) with the momentum p_i being replaced by the velocity v_i. With accuracy up to the first order in $\epsilon \ll 1$ the expansion (19) takes the explicit form

$$\widehat{\mathcal{C}} = \left(M^{(0)}_{\mu\nu} - \gamma_i M^{(1)}_{\mu\nu}\right) g_{\mu i} g_{\nu i} \frac{\partial^2}{\partial v_i^2} + \epsilon M^{(1)}_{\mu\nu} g_{\mu i} g_{\nu i} \left[-\frac{1}{g_{\nu i}} \left(\frac{\partial g_{\nu i}}{\partial x_i}\right) \left(\frac{\partial}{\partial v_i} + v_i \frac{\partial^2}{\partial v_i^2}\right) + \frac{\partial^2}{\partial x_i \partial v_i}\right] + O(\epsilon^2). \tag{28}$$

To obtain the usual probability function $\mathcal{P}(x_i,t)$ we consider velocity moments of the initial distribution function $P(x_i,v_i,t)$ in the standard form [6]

$$\mathcal{P}_n(x_i,t) \equiv \int v_i^n P(x_i,v_i,t)\mathrm{d}v_i \tag{29}$$

where integration over all set $\{v_i\}$ is performed. Then, if we multiply the Fokker–Planck equation (25) by the factor v^n and integrate over velocities, we derive the following recurrent relations:

$$\begin{aligned} &\epsilon^2 \frac{\partial \mathcal{P}_n}{\partial t} + n\gamma_i \mathcal{P}_n + \epsilon \left[\frac{\partial \mathcal{P}_{n+1}}{\partial x_i} - n \left(f_i + \frac{D}{2d} \sum_j \widehat{D}_{ij} x_j \right) \mathcal{P}_{n-1} \right] \\ &= n(n-1) \left(M^{(0)}_{\mu\nu} - \gamma_i M^{(1)}_{\mu\nu}\right) g_{\mu i} g_{\nu i} \mathcal{P}_{n-2} - \epsilon n M^{(1)}_{\mu\nu} \left[g_{\mu i} g_{\nu i} \frac{\partial \mathcal{P}_{n-1}}{\partial x_i} + n g_{\mu i} \left(\frac{\partial g_{\nu i}}{\partial x_i}\right) \mathcal{P}_{n-1} \right] + O(\epsilon^2). \end{aligned} \tag{30}$$

At $n=0$, we obtain the equation for the distribution function $\mathcal{P} \equiv \mathcal{P}_0(x_i,t)$:

$$\frac{\partial \mathcal{P}}{\partial t} = -\epsilon^{-1} \frac{\partial \mathcal{P}_1}{\partial x_i}. \tag{31}$$

The expression for the first moment $\mathcal{P}_1$ follows from Eq.(30) where $n=1$ and only terms of the first order in ϵ are kept:

$$\begin{aligned} \mathcal{P}_1 = \frac{\epsilon}{\gamma_i} \Bigg\{ &\left(f_i + \frac{D}{2d} \sum_j \widehat{D}_{ij} x_j \right) \mathcal{P} \\ -\frac{\partial \mathcal{P}_2}{\partial x_i} &- M^{(1)}_{\mu\nu} \left[g_{\mu i} g_{\nu i} \frac{\partial \mathcal{P}}{\partial x_i} + g_{\mu i} \left(\frac{\partial g_{\nu i}}{\partial x_i}\right) \mathcal{P} \right] \Bigg\}. \end{aligned} \tag{32}$$

The second moment $\mathcal{P}_2$ can be obtained if one puts in Eq.(30) $n=2$ and takes into account only zeroth terms of smallness over the parameter $\epsilon \ll 1$:

$$\mathcal{P}_2 = \left(\frac{M^{(0)}_{\mu\nu}}{\gamma_i} - M^{(1)}_{\mu\nu} \right) g_{\mu i} g_{\nu i} \mathcal{P}. \tag{33}$$

At last, the Fokker–Planck equation takes the Kramers–Moyal form

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial}{\partial x_i} \left(\mathcal{D}_1 \mathcal{P}\right) + \frac{\partial^2}{\partial x_i^2} \left(\mathcal{D}_2 \mathcal{P}\right), \tag{34}$$

where effective drift and diffusion coefficients are as follows:

$$\begin{aligned} \mathcal{D}_1 = \frac{1}{\gamma_i} \Bigg\{ &\left(f_i + \frac{D}{2d} \sum_j \widehat{D}_{ij} x_j \right) \\ &+ \left[M^{(0)}_{\mu\nu} g_{\mu i} g_{\nu i} \frac{\partial \gamma_i^{-1}}{\partial x_i} + M^{(1)}_{\mu\nu} g_{\mu i} \left(\frac{\partial g_{\nu i}}{\partial x_i}\right) \right] \Bigg\}, \end{aligned} \tag{35}$$

$$\mathcal{D}_2 = \frac{M^{(0)}_{\mu\nu}}{\gamma_i^2} \, g_{\mu i} g_{\nu i}. \tag{36}$$

To proceed the consideration we need to pass from the grid representation to a continuous one. In so doing, we use the mean field approximation to replace the second term of effective interaction force in Eq.(35):

$$\frac{D}{2d} \sum_j \widehat{D}_{ij} x_j \to D(\eta - x) \tag{37}$$

where an order parameter $\eta \equiv \langle x \rangle$ is defined through the self–consistency equation

$$\eta = \int_{-\infty}^{\infty} x \mathcal{P}_\eta(x) \mathrm{d}x, \tag{38}$$

$\mathcal{P}_\eta(x)$ is a solution of the Fokker–Planck equation (34). Under stationary condition, the relevant distribution function has the form

$$\mathcal{P}_\eta(x) = \frac{\mathcal{Z}_\eta^{-1}}{\mathcal{D}_2(x)} \exp\left(\int_{-\infty}^{x} \frac{\mathcal{D}_1(x',\eta)}{\mathcal{D}_2(x')} \mathrm{d}x' \right) \tag{39}$$

where the partition function

$$\mathcal{Z}_\eta = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\mathcal{D}_2(x)} \exp\left(\int_{-\infty}^{x} \frac{\mathcal{D}_1(x',\eta)}{\mathcal{D}_2(x')} \mathrm{d}x' \right) \tag{40}$$

takes care of normalization condition. The equation (38) has solutions within the domain bounded by the

Newton–Raphson condition

$$\int_{-\infty}^{\infty} x \frac{\partial}{\partial \eta} \mathcal{P}_\eta(x) \bigg|_{\eta=0} \mathrm{d}x = 1 \tag{41}$$

obtained by differentiating Eq.(38) over the order parameter η.

III. MODEL OF CORRELATION BETWEEN ADDITIVE AND MULTIPLICATIVE NOISES

To apply the general results obtained in Section II we consider in details the simplest model of two correlated noises being additive and multiplicative in nature. Relevant amplitudes are defined as follows:

$$g_a(x) = 1, \quad g_m(x) = \mathrm{sign}(x)|x|^a \tag{42}$$

where the exponent is defined as $a \in [0, 1]$ and the sign function is introduced to take into account a direction of the Langevin force. We will focus on the prototype system concerning the Ginzburg–Landau model with the potential

$$V_0(x) = -\frac{\varepsilon}{2} x^2 + \frac{1}{4} x^4 \tag{43}$$

where ε is a parameter being dimensionless temperature counted off a critical value in negative direction. In correspondence with the line of consideration [14], we take up the viscosity coefficient in the form

$$\gamma(x) = |x^2 - 1|^{-\alpha} \tag{44}$$

where positive index α stands to measure the damping weakening near bare state $x = 1$.

Next, we suppose the noises are to be Gaussian distributed, white in space with zero mean and colored in time according to the correlation matrix

$$\widehat{\mathbf{C}}(\tau) = \begin{pmatrix} \frac{\sigma_a^2}{\tau_a} e^{-|\tau|/\tau_a} & \frac{\sigma_a \sigma_m}{\tau_c} e^{-|\tau|/\tau_c} \\ \frac{\sigma_m \sigma_a}{\tau_c} e^{-|\tau|/\tau_c} & \frac{\sigma_m^2}{\tau_m} e^{-|\tau|/\tau_m} \end{pmatrix} \tag{45}$$

where σ_a and σ_m are amplitudes of additive and multiplicative noises, respectively, τ_a and τ_m are corresponding autocorrelation times, τ_c is a time of the cross–correlation between the noises. Moments (21) of both zero and first orders of correlation matrix (45) are as follows:

$$\begin{aligned} \widehat{\mathbf{M}}^{(0)} &= \begin{pmatrix} \sigma_a^2 & \sigma_a \sigma_m \\ \sigma_m \sigma_a & \sigma_m^2 \end{pmatrix}, \\ \widehat{\mathbf{M}}^{(1)} &= \begin{pmatrix} \tau_a \sigma_a^2 & \tau_c(\sigma_a \sigma_m) \\ \tau_c(\sigma_m \sigma_a) & \tau_m \sigma_m^2 \end{pmatrix}. \end{aligned} \tag{46}$$

Then, the expressions (35), (36) for effective drift and diffusion coefficients take the form

$$\begin{aligned} \mathcal{D}_1 = &|x^2 - 1|^\alpha \left\{ \left[D(\eta - x) + x(\varepsilon - x^2) \right] + a\sigma_m |x|^{a-1} \left[\sigma_a \tau_c + \sigma_m \tau_m \mathrm{sign}(x) |x|^a \right] \right\} \\ &+ 2\alpha\, x (x^2 - 1)^{2\alpha - 1} \left[\sigma_a + \sigma_m \mathrm{sign}(x) |x|^a \right]^2, \end{aligned} \tag{47}$$

$$\mathcal{D}_2 = (x^2 - 1)^{2\alpha} \left[\sigma_a + \sigma_m \mathrm{sign}(x) |x|^a \right]^2. \tag{48}$$

It is worthwhile to notice at $\gamma(x) = \mathrm{const}$ ($\alpha = 0$) the additive noise can not give a contribution to the drift coefficient (47) related to the first of moments (46).

IV. NOISE CORRELATION INDUCED TRANSITIONS

Addressing to the influence of noise cross–correlations on the system behavior we start with self–consistency equation (38) where the stationary distribution (39) is given through the drift and diffusion coefficients (47), (48). It is well known, at phase transitions, the symmetry breaking causes the ordered state corresponding to the solution $\eta \neq 0$ within the interval of available values of stochastic quantity x, while the disordered phase is related to $\eta = 0$. In the absence of the multiplicative noise, the only reason to break the symmetry of the stochastic distribution is the interaction force (37) which plays the role of a conjugate field related to the order parameter η. The principle feature of far-off-equilibrium systems with colored noise is that the symmetry can be restored due to combined effect of both the multiplicative noise and the system nonlinearity [8, 12]. Therefore, the reentrant phase transition in such systems is appeared. Our aim is to demonstrate the course of the phase transition can be crucially changed by means of cross–correlations.

First, we consider the solution of Eq.(38) at different values of the noise cross–correlation scale τ_c. As shown in Figure 1, in the absence of both noises ($\sigma_a = \sigma_m = 0$) and coupling ($D = 0$) the system behaves itself in a usual manner being inherent in the square–root law (dashed curve with both vertical derivative in the point of origin and symmetry with respect to the ε–axis). Such behavior means the maxima appearance of the distribution (39) in points

$\pm\sqrt{\varepsilon}$ that can be interpreted as standard noise induced transition of the second order with mean value $\eta = 0$ [15]. With switching on noises and coupling the situation is changed principally. First, the above symmetry is broken to survive the only negative value of the order parameter in the limit of small cross–correlations (curve 1). Combined effect of correlated noises, system nonlinearity and spatial coupling arrives at the change of the order parameter sign at small values η. Indeed, as seen from curves 2, 3, an increase in the cross–correlation time τ_c shifts weakly negative solution into the positive domain causing the reorientation transition at the driving parameter ε_r. In addition to this transition, an increase in the cross–correlation scale arrives at positive solutions appeared according to discontinuous phase transitions that has reentrant nature within domain bounded by both lower ε_c and upper ε^c boundaries (see elliptic form parts of curves 2, 3 where solid and dotted lines relate to (meta)stable [16] and unstable solutions). With subsequent growth of the cross–correlation time above the value τ_{cr} related to thin solid curve in Figure 1, back bifurcation happens and temperature dependence of the order parameter takes one–connected character. This means an increase in the control parameter ε arrives at the (meta)stable branch of positive magnitudes η initially (solid curve), then the unstable branch (dotted curve) follows from the point ε^c down to ε_c and finally the negative (meta)stable state is merged. It results in formation of a hysteresis loop in $\eta(\varepsilon)$ dependence where both (meta)stable and unstable states exist to be solutions of Eq.(38) (see curves 4, 5). Thus, one can conclude both reentrance and reorientation phase transitions are inherent in systems with colored multiplicative noise.

In Figure 2 we plot a phase diagram in (ε, τ_c) plane to show the influence of the noise cross–correlation scale on the bifurcation magnitudes of the control parameter. It is seen, at small cross–correlation times τ_c, the negative values of the order parameter η is inherent in the whole ε–domain denoted as N in Figure 2. With increase in τ_c the reorientation phase transition occurs into the state P related to the positive solutions $\eta > 0$. The line of this transition is determined by the self–consistency equation (38) at condition $\eta = 0$. At a magnitude τ_0, a doubly bounded domain R appears to relate to the reentrant transition. At the critical value τ_{cr} corresponding to the bifurcation curve in Figure 1 the domain R passes to a region M where all stable, metastable and unstable phases take place. Over the critical correlation time ($\tau_c > \tau_{cr}$), the dotted curve relates to the lower critical value ε_c of the control parameter in Figure 1 (curve 4).

The influence of the multiplicative noise intensity σ_m^2 on the phase transition picture is demonstrated in Figure 3. The upper panel relates to moderate cross–correlation times τ_c where an increase in σ_m^2 transforms two–connected $\eta(\varepsilon)$ dependence into one–connected one varying more fast. With growth of the time scale τ_c, the increase of the multiplicative noise intensity arrives at the shrinking the metastability domain (see lower panel in Figure 3). Thus, we get the conclusion about dual role of the multiplicative noise: at small intensities $\sigma_m^2 \ll 1$, main influence is rendered by the cross–correlations between additive and multiplicative components of the noise to sharpen the phase transition (see Figure 1); on the other hand, a raising the intensity of the latter component up to $\sigma_m^2 \sim 1$ smears this transition.

According to Figure 4 only one difference of the phase diagram in the axes (ε, D) from the situation depicted in Figure 2 takes place. Here, at small magnitudes of the control parameter ε, the domain P of the positive valued order parameter $\eta > 0$ relates to the whole region of the coupling parameter D — contrary to low bounded domain of the cross–correlation times τ_c.

To find relations between noise exponent a and the control parameter ε we consider the phase diagram in (ε, a) plane. It is appeared for noises with weak cross-correlation ($\tau_c \to 0$) the new phase arises only at small enough values of a which define the power of the multiplicative noise (see dashed curve in Figure 5). In other words, considering the class of systems with both additive and multiplicative noises, one should mean that ordering processes are possible in the case of weak cross–correlation only if the multiplicative noise has a weak power. For the systems with $a \to 1$ weak cross–correlation can not induce new phase formation. According to solid–dotted curves in Figure 5, an increase in τ_c leads to appearance of the ε small valued domain where the reorientation transition takes place with a–growth. Besides, domains of both positive and negative order parameters, being reoriented, join with metastable phase region at small values of index a. However, an increase in a leads to the reentrant phase transition for long range cross–correlations. At small and moderate values of the noise exponent a the system behavior is inherent in the hysteresis loop formation.

The influence of the damping exponent $\alpha \neq 0$ on the breaking symmetry picture is shown in Figure 6. It is appeared an increase in α transforms the $\eta(\varepsilon)$ dependence in a manner similar to the influence of the cross–correlation time τ_c. Indeed, passage from two–connected $\eta(\varepsilon)$ dependence related to curves 1 to one–connected curves 2, 3 can be provided with both τ_c increase and α growth — quite similarly to the $\eta(\varepsilon)$ dependencies variation in Figure 1. The conclusion about similarity of the influences of the damping exponent $\alpha \neq 0$ and the cross–correlation time τ_c is confirmed with phase diagram in plane (ε, α) which is topologically identical to the same in Figure 2 at small correlation times (see Figure 7a). According to the Figure 7b the cross–correlation time increase arrives at the reentrant phase transition (within the domain R) due to appearance of additional region N related to negative values of the order parameter.

Finally, we set up the properties of the Langevin sources which allows us to produce the ordering processes in the system. With this aim, we plot the corresponding phase diagram in (τ_c, a) plane (see Figure 8). Here, the system undergoes reorientation transition related to the transforming the negative valued order parameter into the positive one if τ_c increases. On the other hand, at small ε, increasing the exponent a of multiplicative noise, we can make the system to undergo a chain of phase transitions at which the parameter η changes the sign three times as maximum (at $\varepsilon = 6.5$ and $\tau_c = 2.85$ for example, see Figure 8a). The physical situation becomes more simple with an increase in ε (Figure 8b).

V. DISCUSSION

To understand main features of the system under consideration we proceed from equation of effective motion

$$\dot{x} = \mathcal{D}_1(x) + \sqrt{\mathcal{D}_2(x)}\xi(t) \tag{49}$$

related to the Fokker–Planck equation (34). In difference of the initial noises $\zeta_\mu(t)$ in Eq.(2) effective noise $\xi(t)$ is of white–type: $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(0)\rangle = \delta(t)$. Within the mean field approach, Eq.(49) takes the form

$$\dot{\eta} = \langle\mathcal{D}_1(x)\rangle \simeq -\frac{\partial F}{\partial\eta}, \quad F \equiv \Delta F(\eta) - h\eta. \tag{50}$$

With accounting definition (47), where the simplest set of indexes $\alpha = 0$, $a = 1$ is chosen, a thermodynamic–type potential $\Delta F(\eta)$ and a field h are defined by the following expressions:

$$\Delta F(\eta) = -\frac{\varepsilon + \varepsilon_m}{2}\eta^2 + \frac{1}{4}\eta^4, \quad \varepsilon_m \equiv \tau_m\sigma_m^2; \tag{51}$$

$$h \equiv \tau_c\sigma_a\sigma_m. \tag{52}$$

Comparison of the first of definitions (51) with the bare potential (43) shows the multiplicative noise arrives at increase of the control parameter ε due to addition ε_m whose magnitude is proportional to the noise intensity σ_m^2 with the coefficient τ_m being self–correlation time. As a result, a growth of the multiplicative noise intensity in Figure 3 causes increasing the order parameter η at small magnitudes of the control parameter ε. A smearing of the related dependencies $\eta(\varepsilon)$ at moderate values ε is caused by effective field h.

This field is inherent in the cross–correlation effect fixed by the characteristic time τ_c and intensities σ_a, σ_m of both additive and multiplicative noises. According to Eq.(50) the field h leads to deepening the right minimum of the thermodynamic potential $F(\eta)$. If cross–correlation effects are so slight that the condition

$$\tau_c < C\,\frac{(\varepsilon + \varepsilon_m)^{3/2}}{\sigma_a\sigma_m}, \quad C \equiv \frac{2}{3^{3/2}} \simeq 0.385 \tag{53}$$

is applied, the field h is less than a critical value $h_c = C(\varepsilon + \varepsilon_m)^{3/2}$ and the right minimum of the thermodynamic potential $F(\eta)$ has a local character. It means the positive order parameter appears within a two–bounded interval $\varepsilon_c < \varepsilon < \varepsilon^c$ (see curves 2, 3 in Figure 1). With strengthening cross–correlations, when the condition $h < h_c$ ceases to be valid, a barrier between right and left minima disappears and a domain of definition of the positive order parameter becomes bounded by the only upper boundary ε^c (curves 4, 5 in Figure 1).

With passage to the general case $a \neq 1$, the thermodynamic potential in Eqs.(50), (51) takes the form

$$F = \Delta F(\eta) - h\ \mathrm{sign}(\eta)|\eta|^a, \tag{54}$$

$$\Delta F = -\left(\frac{\varepsilon}{2}\eta^2 + \frac{\varepsilon_m}{2}\eta^{2a}\right) + \frac{1}{4}\eta^4 \tag{55}$$

that differs from the initial one by replacement $\eta \to \mathrm{sign}(\eta)|\eta|^a$. As $a \leq 1$, this replacement derives to more strong variations of the thermodynamic potential $V(\eta)$ within the actual domain $\eta < 1$ that can arrive at the appearance of local minimum at moderate values η. As a result, a decrease of the index a derives to metastable phase — in perfect accordance with Figure 5.

To ascertain the effect of the index α we consider mean field approach in the extreme case $\alpha = 1$, $a = 1$. Here, the thermodynamic potential in the equation of motion (50) takes the form

$$F = F_\lessgtr \mp h\eta + \Delta F(\eta), \quad F_\lessgtr \equiv \frac{1}{6} - \frac{\varepsilon + \varepsilon_m}{2} \pm \frac{4}{3}h \tag{56}$$

where reference points $F_\lessgtr$ correspond to the domains $\eta < -1$ and $\eta > 1$, respectively, the field h is determined by Eq.(52) and the addition $\Delta F(\eta)$ is defined by the expansion

$$\Delta F \equiv \frac{A}{2}\eta^2 + \frac{B}{3}\eta^3 + \frac{C}{4}\eta^4 + \frac{E}{5}\eta^5 + \frac{G}{6}\eta^6 \tag{57}$$

with the following coefficients:

$$\begin{aligned} &A \equiv \mp(\varepsilon + \varepsilon_m) + 2\sigma_a^2, \quad B \equiv (4 \pm \tau_c)\sigma_a\sigma_m, \\ &C \equiv \pm\left[1 + (\varepsilon + \varepsilon_m)\right] + 2(\sigma_m^2 - \sigma_a^2), \\ &E \equiv -4\sigma_a\sigma_m, \quad G \equiv \mp 1 - 2\sigma_m^2; \end{aligned} \tag{58}$$

the upper and lower signs $\pm$ in the first Eq.(56) and Eqs.(58) relate to the domains $|\eta| < 1$ and $|\eta| > 1$, respectively. Comparing these equations with the potential (51) addressed to the index $\alpha = 0$ we convince the dispersion of the damping coefficient (44) arrives at transformation of the second order phase transition into the first one — as it follows from Figures 6, 7.

The form of the thermodynamic potential given by Eqs.(56) — (58) is shown in Figure 9 as a function of the order parameter. It is seen this dependence has three well pronounced minima to be inherent in the first order phase transition. Moreover, comparison of the curves 1 and 2 shows the switching on the interaction field (37) arrives at the gradient of the dependence $F(x)$ inducing the break symmetry. On the other hand, if we would like to pass from the Ito calculus above used to the Stratonovich one, we have to add the term $\frac{1}{2}\mathcal{D}_2'(x)$ to the drift coefficient $\mathcal{D}_1(x)$ in Eq. (50) [4]. Then, the thermodynamic potential $F \equiv -\int \mathcal{D}_1(\eta)\mathrm{d}\eta$ obtains the addition $-\frac{1}{2}\mathcal{D}_2(\eta)$ to transform the potential given by Eqs.(56) — (58) to the form

$$\begin{aligned}
&\widetilde{F} = \widetilde{F}_{\lessgtr} - \tilde{h}\eta + \Delta\widetilde{F}(\eta);\\
&\widetilde{V}_{\lessgtr} \equiv \frac{1}{6} - \frac{\varepsilon + (\sigma_a^2 + \tau_m\sigma_m^2)}{2} \pm \frac{4}{3}\tau_c\sigma_a\sigma_m,\\
&\tilde{h} \equiv (1 \pm \tau_c)\sigma_a\sigma_m,\\
&\Delta\widetilde{F} \equiv \frac{\tilde{A}}{2}\eta^2 + \frac{\tilde{B}}{3}\eta^3 + \frac{\tilde{C}}{4}\eta^4 + \frac{\tilde{E}}{5}\eta^5 + \frac{\tilde{G}}{6}\eta^6,\\
&\tilde{A} \equiv \mp\varepsilon + \left[4\sigma_a^2 - (1 \pm \tau_m)\sigma_m^2\right],\\
&\tilde{B} \equiv (10 \pm \tau_c)\sigma_a\sigma_m,\\
&\tilde{C} \equiv \pm(1+\varepsilon) - \left[4\sigma_a^2 - (6 \pm \tau_m)\sigma_m^2\right],\\
&\tilde{E} \equiv -9\sigma_a\sigma_m, \quad \tilde{G} \equiv \mp 1 - 5\sigma_m^2
\end{aligned} \tag{59}$$

where we take into account Eq.(48) at $\alpha = 1$, $a = 1$. Comparison of the corresponding dependencies $\widetilde{F}(\eta)$ shown in insertion of Figure 9 with initial ones $F(\eta)$ depicted in the main panel shows the Stratonovich addition promotes to transformation of the first order transition into the second one.

We proceed with consideration of the form of the probability distribution function (39) that is responsible for the reorientation transition related to curve 1 in Figure 3b. As shows comparison of the curves α and ϵ in the Figure 10a, positive magnitudes of the order parameter $\eta > 0$ is related to the distribution whose right maximum has a larger height and is a wider than the left one (and vice versa at $\eta < 0$). Much more complicated picture takes place with growth of the correlation time τ_c when strongly pronounced maximum of the distribution (39) is transformed from the left into the right one by means of passage via the bimodal dependence (see curves β, γ and δ in Figure 10b).

Above considered situation is picked out to address to the constant damping coefficient (44) when the distribution (39) has smooth form due to the index $\alpha = 0$. With passage to general case $\alpha \neq 0$ the dependence $\mathcal{P}_\eta(x)$ obtains the pair of the strong maxima in symmetrical points $x = \pm 1$ (see Figure 11). The analytical form of these maxima follows from the estimations

$$\begin{aligned}
&\mathcal{D}_1(x) \simeq 2\alpha\left(\sigma_a \pm \sigma_m\right)^2 x(x^2-1)^{2\alpha-1},\\
&\mathcal{D}_2(x) \simeq \left(\sigma_a \pm \sigma_m\right)^2 |x^2-1|^{2\alpha}
\end{aligned} \tag{60}$$

that are given by the dependencies (47), (48) near the points $x = \pm 1$. As a result, we arrive at the integrable singularities

$$\mathcal{P}_\eta(x) \simeq \frac{\mathcal{Z}_\eta^{-1}}{\left(\sigma_a \pm \sigma_m\right)^2 |x^2-1|^{\alpha}} \tag{61}$$

which have the form of the maxima shown in Figure 11. In contrary, near the point $x = 0$ one has the estimations

$$\mathcal{D}_1(x) \simeq a\sigma_a\sigma_m\tau_c|x|^{a-1}, \quad \mathcal{D}_2(x) \simeq \sigma_a^2 \tag{62}$$

which arrive at the expression

$$\mathcal{P}_\eta(x) \simeq \frac{\mathcal{Z}_\eta^{-1}}{\sigma_a^2}\exp\left(\frac{\sigma_m}{\sigma_a}\tau_c|x|^a\right) \simeq \frac{\mathcal{Z}_\eta^{-1}}{\sigma_a^2}. \tag{63}$$

Thus, the singularities of the drift coefficient $\mathcal{D}_1(x)$ at the point of origin has integrable character to derive to the finite value of the probability distribution function.

Traditionally, one is taken to present the distribution function (39) in the Boltzmann–Gibbs exponential form

$$\mathcal{P}(x) \equiv \exp\left\{-\frac{V_{ef}(x)}{\sigma_a^2}\right\} \tag{64}$$

where an effective potential

$$V_{ef} \equiv -\sigma_a^2\int\frac{\mathcal{D}_1(x)}{\mathcal{D}_2(x)}\mathrm{d}x + \sigma_a^2\ln\mathcal{D}_2(x) \tag{65}$$

is introduced to govern by the probability distribution in the usual manner [1]. Usage of the definitions (47) and (48) in the simplest case $a = 1$, $\alpha = 0$ derives to explicit form of the potential (65):

$$
\begin{aligned}
V_{ef} = &- \mathcal{H}x + \mathcal{V}(x); \qquad \mathcal{H} \equiv D\eta + (\tau_c - 2)\sigma_a\sigma_m, \quad \mathcal{V} \equiv \frac{\mathcal{A}}{2}x^2 + \frac{\mathcal{B}}{3}x^3 + \frac{\mathcal{C}}{4}x^4, \\
&\mathcal{A} \equiv (D-\varepsilon) - (2+\tau_m)\sigma_m^2 + 2\frac{\sigma_m}{\sigma_a}(D\eta + \tau_c\sigma_a\sigma_m), \\
&\mathcal{B} \equiv 2\frac{\sigma_m}{\sigma_a}\left[(\varepsilon - D) + (1+\tau_m)\sigma_m^2\right] - 3\left(\frac{\sigma_m}{\sigma_a}\right)^2 (D\eta + \tau_c\sigma_a\sigma_m), \\
&\mathcal{C} \equiv 1 - \left(\frac{\sigma_m}{\sigma_a}\right)^2 \left[3(\varepsilon - D) + (2+3\tau_m)\sigma_m^2\right] + 4\left(\frac{\sigma_m}{\sigma_a}\right)^3 (D\eta + \tau_c\sigma_a\sigma_m)
\end{aligned}
\tag{66}
$$

where we kept only the terms up to the fourth order of the stochastic variable x. The form of the dependence (65) is depicted in Figure 12 for different sets of the indexes a and α. Comparison of the respective curves confirms the growth of the index α promotes to strengthening pair of the strong minima at points $\eta = \pm 1$ of the bias dependence $V_{ef}(x)$.

VI. CONCLUSIONS

In this paper we have considered the effect of the ordering of stochastic system with two correlated noises. In so doing, we have used a model of the system with Landau–like potential $V_0(x)$, subject to both additive and multiplicative noises with amplitude of the last in the form of the power–law function $|x|^a$, $a \in [0,1]$ and affected by x–dependent damping with coefficient $\gamma(x) = |x^2 - 1|^{-\alpha}$, $\alpha \in [0,1]$. Within the framework of both the cumulant expansion method and mean field theory, the stationary picture of the ordered states is investigated in details. We have shown the fluctuation cross–correlations can lead to the symmetry breaking of the distribution function even in the case of the zero–dimensional system. With introducing the spatial coupling, noise cross–correlations can induce phase transitions where the order parameter $\eta = \langle x \rangle$ varies discontinuously or in reentrant manner. We have studied the specified interval of magnitudes of system parameters where the ordered phase can be formed. With this aim, principle phase diagrams are obtained to illustrate the role of the multiplicative noise exponent a, spectral characteristics of fluctuations (auto–correlation time τ_m and cross–correlation time τ_c), amplitudes of both additive σ_a and multiplicative σ_m noises, exponent α of the kinetic coefficient $\gamma(x)$, as well as deterministic parameters being the dimensionless temperature ε and intensity of the spatial coupling D.

Above studied picture allows us to generalize the theory of the phase transitions to the system with set of stochastic forces of different nature. Basing on the mean field approach we have shown the system can be described through the thermodynamic potential $F(\eta)$ whose construction differs principally from the bare potential $V_0(x)$: so, if the latter has the simplest x^4–form, the former is shown to be of the η^6–form. Coefficients of related expansion are obtained to define terms of the even powers through the dimensionless temperature ε, intensity of spatial coupling D, auto–correlation time τ_m and intensities of both additive σ_a^2 and multiplicative σ_m^2 noises; terms of the odd powers are defined through the characteristics τ_c, σ_a and σ_m of the noise cross–correlations, respectively. Thus, we can conclude the phase transition tends to transform its character from continuous to discontinuous due to the noise cross–correlation strengthening. This trend displays more strongly with growth of the index α whose value determines the dispersion of the damping coefficient $\gamma(x)$. On the other hand, transition from the Ito calculus to the Stratonovich one promotes to inverse transformation of the discontinuous transition into the continuous one.

Obtained results can be applied to a consideration of the complex systems which are far-off-equilibrium and hold several collective degrees of freedom. As shows consideration of three–dimensional Lorenz–like system with noises being initially additive in nature, usage of the saving principle reduces two of these noises to multiplicative ones [17]. The physical reason of such a picture is hierarchical subordination of different degrees of freedom. According to our previous considerations [18], [19] a typical example of such type takes place in solid state physics where a reentrant metastable phase can appear if the matrix phase relates to random ensemble of defects of different dimensions subject to the field of plastic flow (driven dislocation–vacancy ensemble). Here, in the course of plastic flow different defect structures alternate one another according to picture of the first order phase transition. Moreover, structural reorientation transitions take place where the sign of the order parameter is related to the resulting direction of the Burgers vectors of dislocation cluster. One more example of above studied behavior gives reentrant glass transition in colloid–polymer mixtures [20].

Note finally all presented results have been derived for the system with nonconserved order parameter. The perspective of further exploration is to investigate the system with conserved order parameter.

VII. ACKNOWLEDGEMENTS

A. I. O. is gratefully acknowledged STCU, project 1976, for financial support.

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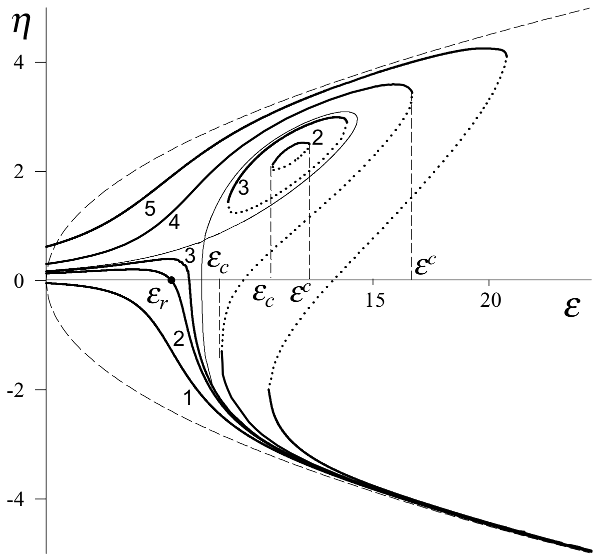

FIG. 1: Dependence of the order parameter η on the control parameter ε at $a = 0.8$, $\alpha = 0$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $\tau_m = 0.01$, $D = 1.0$. Curves 1, 2, 3, 4, 5 correspond to $\tau_c \to 0$, $\tau_c = 2.5$, 3.0, 5.0, 10.0, respectively. Dashed curve relates to bare dependence $\eta = \pm\sqrt{\varepsilon}$, dotted curves correspond to unstable solutions.

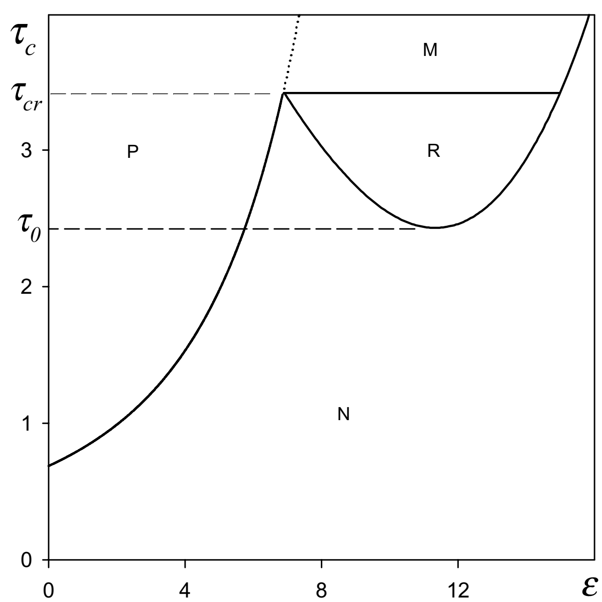

FIG. 2: Phase diagram in (ε, τ_c) plane at $a = 0.8$, $\alpha = 0$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $\tau_m = 0.01$, $D = 1.0$. Domains denoted as P, N, R and M correspond to positive and negative η values, reentrant transition and (meta)stable phases, respectively.

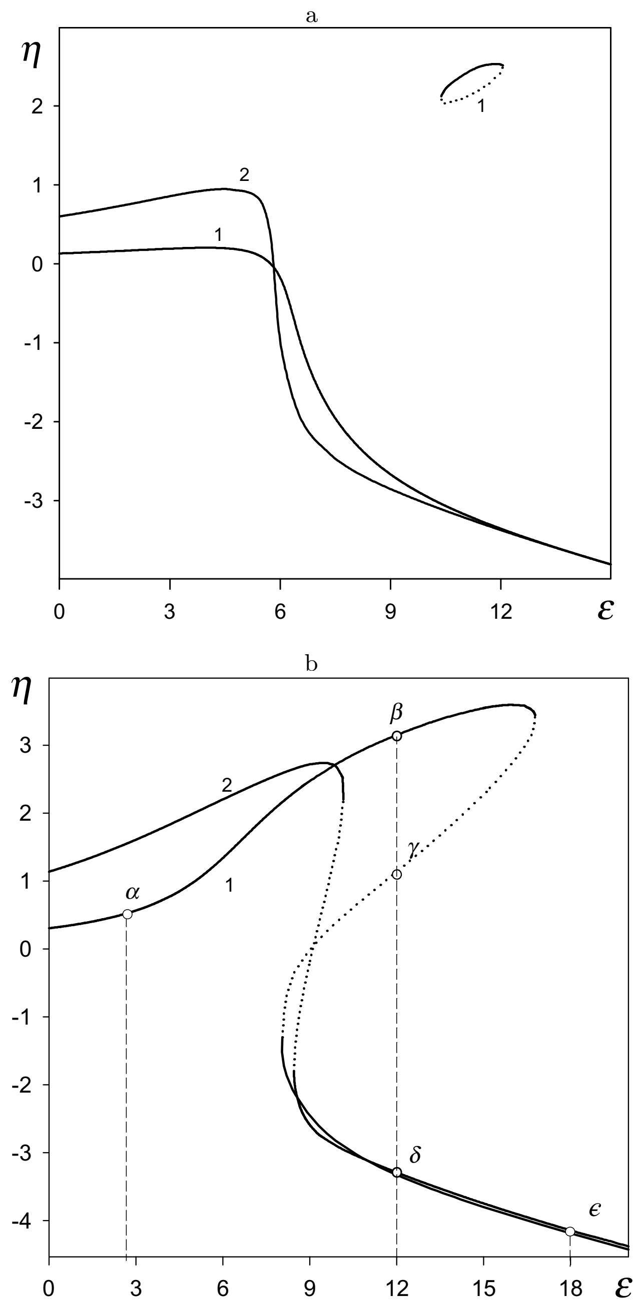

FIG. 3: The order parameter η vs. the control parameter ε at $a = 0.8$, $\alpha = 0.0$, $D = 1.0$, $\sigma_a^2 = 4.84$, $\tau_m = 0.01$ and cross–correlation times $\tau_c = 2.5$ (a) and $\tau_c = 5.0$ (b). Curves 1, 2 relates to the multiplicative noise intensities $\sigma_m^2 = 0.01$ and $\sigma_m^2 = 0.25$. Points from α to ϵ address to corresponding curves in Figure 10.

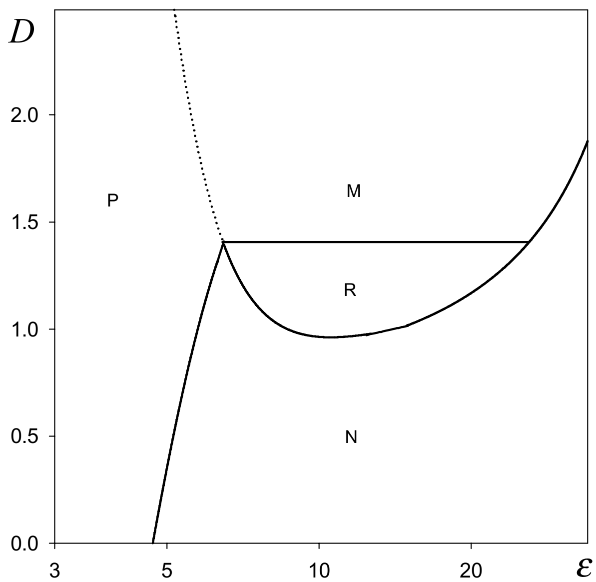

FIG. 4: Phase diagram in (ε, D) plane at $a = 0.8$, $\alpha = 0$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $\tau_m = 0.01$, $\tau_c = 2.5$.

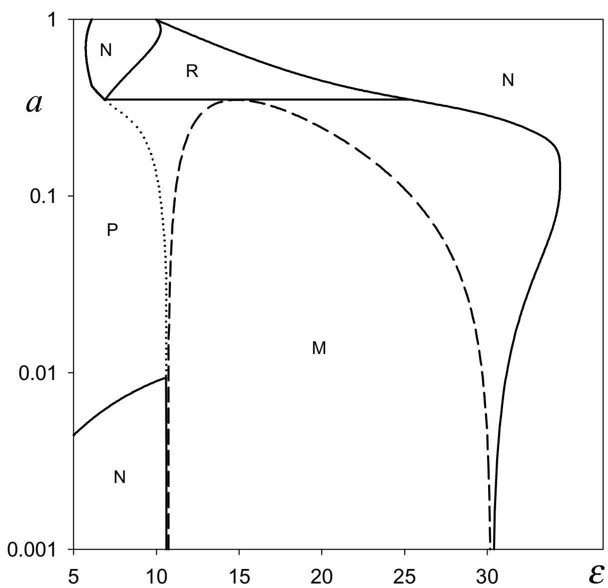

FIG. 5: Phase diagram in (ε, a) plane at $\alpha = 0$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $\tau_m = 0.01$, $\tau_c = 2.5$, $D = 1.0$. Dashed curve relates to the limit $\tau_c \to 0$.

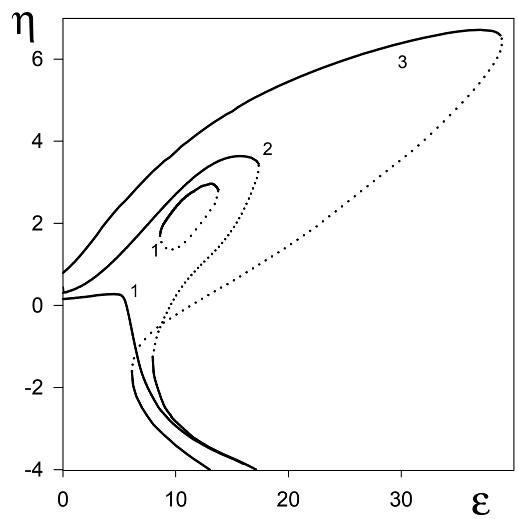

FIG. 6: The order parameter η vs. the control parameter ε at $\tau_m = 0.01$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $D = 1.0$, $a = 0.8$: curve 1 — $\alpha = 0.2$, $\tau_c = 2.5$; curve 2 — $\alpha = 0.2$, $\tau_c = 5.0$; curve 3 — $\alpha = 0.7$, $\tau_c = 5.0$.

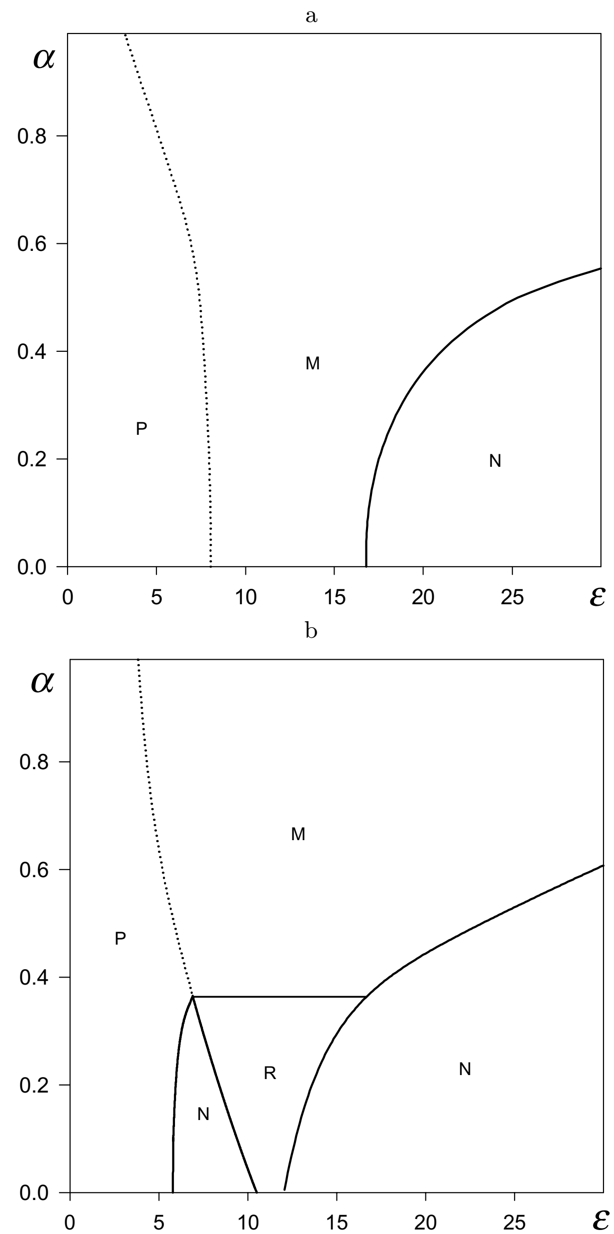

FIG. 7: Phase diagram (ε, α) plane at $\tau_m = 0.01$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $D = 1.0$, $a = 0.8$: a) $\tau_c = 2.5$; b) $\tau_c = 5.0$.

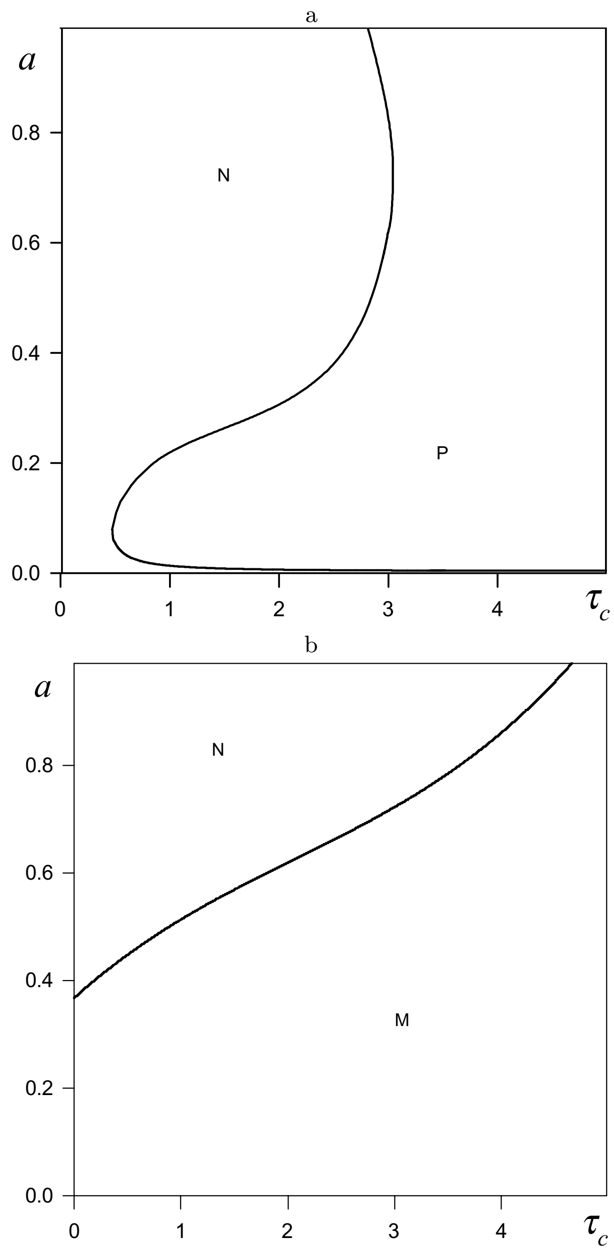

FIG. 8: Noise exponent a vs. noise cross-correlation scale τ_c at $\varepsilon = 6.5$ (a) and $\varepsilon = 15$ (b); other parameters are: $\alpha = 0$, $\tau_m = 0.01$, $\sigma_a^2 = 4.84$, $\sigma_m^2 = 0.01$, $D = 1.0$.

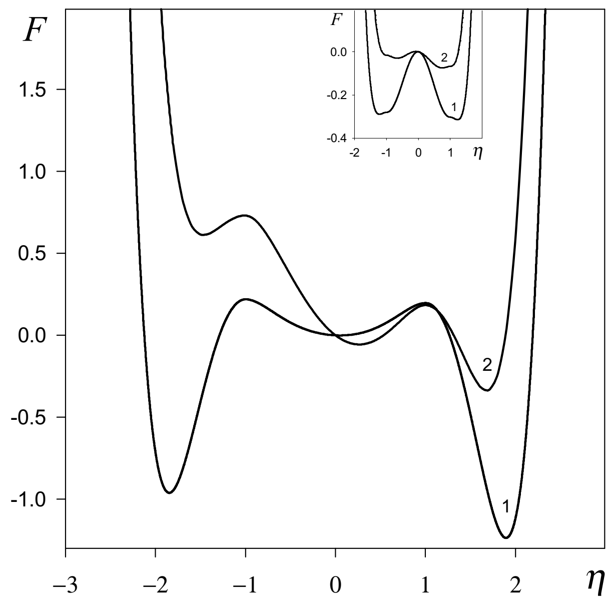

FIG. 9: The form of the thermodynamic potential F given by Eqs.(56) — (58) (insertion shows the potential (59), curves 1 and 2 correspond to $D = 0$ and $D = 1$).

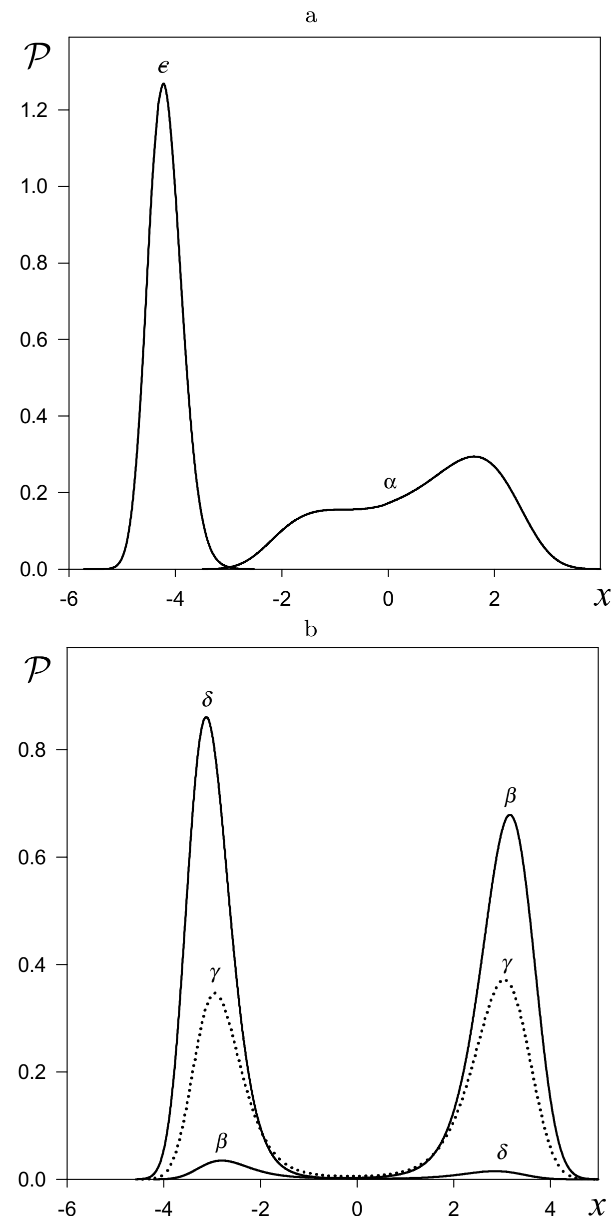

FIG. 10: Probability distributions addressed to different points of the dependence $\eta(\varepsilon)$ related to curve 1 in Figure 3b: curves α, ϵ in panel (a) correspond to the values $\varepsilon = 3$, $\eta = 0.56$ and $\varepsilon = 18$, $\eta = -4.18$, respectively; in panel (b) the control parameter is equal $\varepsilon = 12$ and curves β, γ, δ correspond to different magnitudes of the order parameter $\eta = 3.15$, $\eta = 1.14$ and $\eta = -3.32$.

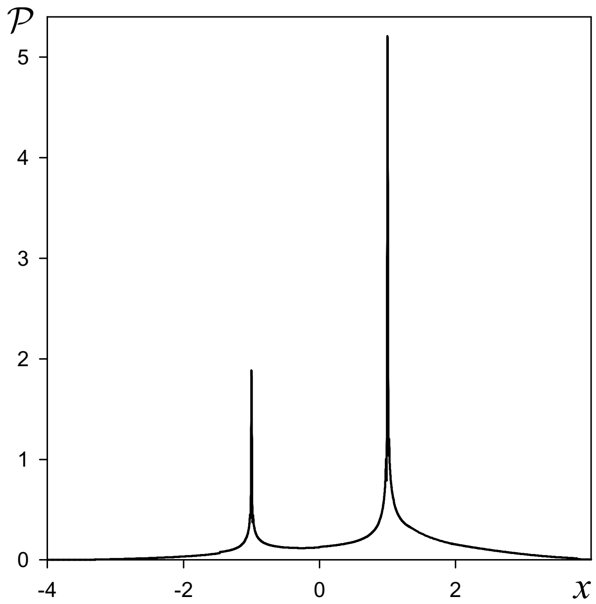

FIG. 11: Probability distribution related at $a = 0.5$, $\alpha = 0.5$, $\varepsilon = 2$.

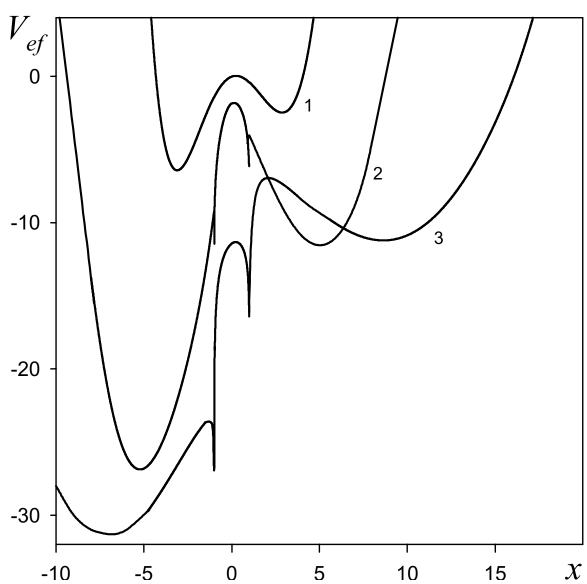

FIG. 12: The form of the effective potential (65): curves 1, 2 and 3 correspond to the cases $a = 1$, $\alpha = 0$ ($\varepsilon = 10$, $\eta = -2.8$); $a = 0.5$, $\alpha = 0.52$ ($\varepsilon = 10$, $\eta = -3.0$) and $a = 1$, $\alpha = 1$ ($\varepsilon = 10$, $\eta = -4.88$), respectively.